MATH4050 Real Analysis
Homework 1 A

There are ~~8~~ 10 questions in this assignment (your works on the questions with * will be marked). The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. * (3rd: P.12, Q6)
Let $f : X \to Y$ be a mapping of a nonempty space X into Y. Show that f is one-to-one if and only if there is a mapping $g : Y \to X$ such that $g \circ f$ is the identity map on X, that is, such that $g(f(x)) = x$ for all $x \in X$.

2. (3rd: P.12, Q7)
Let $f : X \to Y$ be a mapping of X into Y. Show that f is onto if there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map in Y, that is, such that $f(g(y)) = y$ for all $y \in Y$.

3. Show that any set X can be "indexed": $\exists$ a set I and a function $f: I \to X$ such that $\{f(i) : i \in I\} = X$.

4. * (3rd: P.16, Q14)
Given a set B and a collection of sets $\mathcal{C}$. Show that

$$B \cap \left[\bigcup_{A \in \mathcal{C}} A\right] = \bigcup_{A \in \mathcal{C}} (B \cap A).$$

5. (3rd: P.16, Q15)
Show that if $\mathcal{A}$ and $\mathcal{B}$ are two collection of sets, then
$[\bigcup\{A : A \in \mathcal{A}\}] \cap [\bigcup\{B : B \in \mathcal{B}\}] = \bigcup\{A \cap B : (A, B) \in \mathcal{A} \times \mathcal{B}\}$.

6. * (3rd: P.16, Q16)
Let $f : X \to Y$ be a function and $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of X.

   a. Show that $f[\bigcup A_\lambda] = \bigcup f[A_\lambda]$.
   b. Show that $f[\bigcap A_\lambda] \subset \bigcap f[A_\lambda]$.
   c. Give an example where $f[\bigcap A_\lambda] \neq \bigcap f[A_\lambda]$.

7* (3rd: P.16, Q17)
Let $f : X \to Y$ be a function and $\{B_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of Y.

   a. Show that $f^{-1}[\bigcup B_\lambda] = \bigcup f^{-1}[B_\lambda]$.
   b. Show that $f^{-1}[\bigcap B_\lambda] = \bigcap f^{-1}[B_\lambda]$.
   c. Show that $f^{-1}[B^c] = (f^{-1}[B])^c$ for $B \subset Y$.

8. (3rd: P.16, Q18)

a. Show that if f maps X into Y and $A \subset X$, $B \subset Y$, then

$$f[f^{-1}[B]] \subset B$$

and

$$f^{-1}[f[A]] \supset A.$$

b. Give examples to show that we need not have equality in each of the displayed two lines.

c. Show that if f maps X onto Y and $B \subset Y$, then

$$f[f^{-1}[B]] = B.$$

9.* Show that $f \mapsto \int_0^1 f(x)dx$ is a "monotone" function on $R[0,1]$ (consisting of all Riemann integrable functions on $[0,1]$), and $R[0,1]$ is a linear space. Show further that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

if $f, f_n \in R[0,1]$ such that

$$\lim_{n\to\infty} (\sup_{x\in[0,1]} |f_n(x) - f(x)|) = 0.$$

10*: Let $f: [a,b] \to \mathbb{R}$, and $x_0 \in [a,b]$. Show that f is cts (continuous) at x_0 iff

$$f(x_0) = \inf_{\delta>0} \sup\{f(x) : x \in [a,b] \cap V_\delta(x_0)\}$$
$$= \sup_{\delta>0} \inf\{f(x) : x \in [a,b] \cap V_\delta(x_0)\}.$$

(the last two lines are sometimes denoted by $\bar{f}(x_0) \doteq \inf_{\delta>0} f^\delta(x_0)$ and $\underline{f}(x_0) = \sup_{\delta>0} f_\delta(x_0)$ respectively), where

$$V_\delta(x_0) := \{x \in \mathbb{R} : |x - x_0| < \delta\} = (x_0 - \delta, x_0 + \delta).$$